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RESONANT FREQUENCIES OF THE SCATTERING OF ELASTIC WAVES BY THREE-DIMENSIONAL CRACKS†

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The diffraction of elastic waves by three-dimensional cracks of arbitrary cross-section is investigated. Using an earlier method [1] for solving the resulting systems of integral equations, the dependence of the resonance scattering frequencies on the crack shape is analysed. The numerical results indicate that the crack shape has a noticeable effect on the distribution of resonance poles in the complex frequency plane. In theory, this means that the information obtained can be used to solve inverse problems of flaw detection (determining the size and shape of a crack from the reflected signal). © 1999 Elsevier Science Ltd. All rights reserved.

One of the basic problems in ultrasonic flow detection in materials, compounds and structures is to determine the size and shape of a flaw (and a crack, in particular) from the reflected wave field. If the transmitted waves are considerably shorter than the dimensions of the flaw, the problem can be solved in the ray approximation of geometrical optics) [2]. However, in many cases of practical interest, the diameter of the flaw is comparable with or less than the wavelength and the reflected signal gives a "blurred" image, from which its size and shape cannot be determined immediately. Thus in order to establish the shape of the flaw, one must employ special algorithms which minimize a certain specific discrepancy function when the size and shape of the reflecting object are varied [3].

Since the direct diffraction problem is solved at each step of the minimization, the solution of the inverse problem often involves considerable computer costs. More importantly, these problems, in general, are ill-posed, needing special regularization to achieve stable numerical convergence [4]. As we know, one way of regularizing inverse problems is to restrict the set on which the solution is sought (in the case here, the set of allowable shapes for the crack). This can be done using further information on the expected properties of the solution. In the case of inverse wave problems, this could be information on the resonance properties of the object under investigation.

The main parameters used in ultrasonic flaw detection are the time at which the echo signal arrives (to locate the position of the object), the scattering diagrams and the energy reflection coefficient (to determine its orientation, size and shape). The energy reflection coefficient $\Sigma = E_1/E_0$, where E_1 is the energy of the scattered field and E_0 is the energy transferred by the incident wave across an area of the same size as the crack, depends, like the scattering diagram, on the crack shape, the form of the incident wave, the angle of incidence and the frequency. For practical values of the angular frequency ω of steady oscillations $\mathbf{u}e^{-i\omega t}$ the values of $\Sigma(\omega)$ are finite. However, if the solution is continued analytically into the complex plane of ω, $\Sigma(\omega)$ increases without limit at some points ω_k ($k = 1, 2, \ldots$) of the lower half-plane. An important point is that the quantities ω_k, called complex resonance poles or characteristic scattering frequencies, being points of the spectrum of the corresponding integral operators (cf. [1]), depend only on the properties of the material and the region Ω occupied by the crack, and not on the angle of incidence (orientation) or (with some qualifications) on the form of the incident wave. Thus, the amount of information needed to identify the crack shape from the scattered field can be considerably reduced by: (1) first obtaining the distribution of poles as a function of the shape, and (2) being able to separate the resonance frequencies from the recorded reflected signal.

These ideas are the basis of the singularity expansion method (SEM), which was originally devised to locate objects by means of electromagnetic waves [5]. During the 1970s and 1980s various ways of using resonance scattering frequencies of scalar electromagnetic and acoustic wave fields to determine the shape and properties of a scatterer were proposed (see the review in [6]). The method was later applied to elastodynamics [7], but the success achieved was much more modest, primarily because of the difficulty of solving the corresponding direct elastodynamics problems. Thus, the resonance poles in diffraction by three-dimensional cracks have been found only for circular [8] and near-circular (elliptical) [9, 10] cracks. The method that we used in [1] permits quite a wide variation in the crack shape. The results presented below demonstrate how the pole distribution pattern changes when the crack is far from circular.

Consider the diffraction of a given wave field $\mathbf{u}_0(\mathbf{x})e^{-i\omega t}$ by a crack in an elastic space occupying a plane region Ω. The problem reduces to a system of two-dimensional Wiener–Hopf integral equations over Ω with hyper-singular kernel. The method for solving these equations and the notation used are

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given in [1]. The values of the required resonance poles ω_κ of the reflected field $\mathbf{u}_1(\mathbf{x}, \omega)$ can obviously be approximated by the values of the parameter ω for which $\Delta(\omega) = \det A(\omega)$, the determinant of the matrix of the system of linear algebraic equations which arises as a result of discretization (system (1.6) of [1])

$$A(\omega)\mathbf{c}(\omega) = \mathbf{f}(\omega) \tag{1}$$

vanishes.

The corresponding characteristic scattering forms $\mathbf{u}_{1k}(\mathbf{x}, \omega_\kappa)$ are then defined in terms of the eigenvectors $\mathbf{c}_k$: $A(\omega_\kappa)\mathbf{c}_k = 0$. Like the resonance poles, these are independent of the cause of wave emission (of the right-hand side $\mathbf{f}$), and so the information on the characteristic scattering forms can also be used to identify the object. It has also been reported that a rigorous proof has been obtained of the one-to-one correspondence between the size and shape of a crack on the one hand and the first resonance pole and corresponding scattering shape on the other.† It should be noted that not all the zeroes of the determinant $\Delta(\omega)$ are poles of the solution $\mathbf{c}(\omega)$ of system (1) or, therefore, poles of $\mathbf{u}_1(\mathbf{x}, \omega)$. Calculations have shown that in the solution $c^{(n)}(\omega) = \Delta^{(n)}(\omega)/\Delta(\omega)$ $(n = 1, 2, \ldots, N)$ more than half are removed identically by the equal zeros of the minors appearing in $\Delta^{(n)}(\omega)$.

We know that the solution of the harmonic problem for real ω has the property $\mathbf{u}_1(\mathbf{x} - \omega) = \mathbf{u}_1^*(\mathbf{x}, \omega)$ (the asterisk here and henceforth is used to denote the complex conjugate). This property ensures that the solution $\mathbf{v}_1$ of the corresponding transient problem is real-valued when represented in the form of a Laplace–Fourier integral (frequency expansion)

$$\mathbf{v}_1(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{u}_1(\mathbf{x},\omega)e^{-i\omega t}d\omega = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} \mathbf{u}_1(\mathbf{x},\omega)e^{-i\omega t}d\omega \tag{2}$$

In an analytic continuation into the complex plane of ω, this property takes the form

$$\mathbf{u}_1(\mathbf{x},\tilde{\omega}) = \mathbf{u}_1^*(\mathbf{x},\omega) \tag{3}$$

The tilde denotes the transition to complex values, which are symmetric about the imaginary axis: $\omega = -\omega^*$ [11]. It follows from (3) that the poles of $\mathbf{u}_1(\omega)$ are situated symmetrically in the lower half-plane $\operatorname{Im}\omega \leqslant 0$ relative to the imaginary axis $\operatorname{Re}\omega = 0$. There cannot be any poles in the upper half-plane, since they would violate the causality principle. Thus, upwards closure of the integration contour in (2) for $t < t_0$, where t_0 is the time at which the leading front of the reflected field arrives at the point $\mathbf{x}$, would give non-zero values of $\mathbf{v}$ in the form of the sum of residues at these poles. Henceforth we will use ω_k to denote only irremovable poles lying in the right-hand lower quadrant $\operatorname{Re}\omega > 0$, $\operatorname{Im}\omega \leqslant 0$.

For $t > t_0$ the closure of the integration contour in representation (2) into the lower half-plane gives a solution in which there is a contribution from residues and from the integral over the sides of cuts L (if there are branch points)

$$\mathbf{v}_1(t) = -i\sum_k [\operatorname{res}\mathbf{u}_1(\omega)|_{\omega=\omega_k} e^{-i\omega_k t} + \operatorname{res}\mathbf{u}_1(\omega)|_{\omega=\tilde{\omega}_k} e^{-i\tilde{\omega}_k t}] - \frac{1}{2\pi}\int_L \mathbf{u}_1(\omega)e^{-i\omega t}d\omega =$$

$$= 2\sum_k \operatorname{Im}[\operatorname{res}\mathbf{u}_1(\omega)|_{\omega=\omega_k} e^{-i\omega_k t}] - \frac{1}{2\pi}\int_L \mathbf{u}_1(\omega)e^{-i\omega t}d\omega \tag{4}$$

Generally speaking, the integrand in (2) can include exponential components of the form $e^{-i\omega(t-t_n)}$, $t_n > t_0$, which are decreasing from $t_0 < t < t_n$ in the upper half-plane. Thus, in the interval $t_0 < t < \max t_n$ (in physical terms, before the signal has arrived from all points of the radiating object) the contour is closed downwards for only some of the components of $\mathbf{u}_1(\omega)$. Examples are given of the construction of a transient solution in this case in [11] and the problem will not be considered further here.

Allowing for the contribution of symmetric poles ω_k in (4), we have used the properties $\operatorname{res}\mathbf{u}(\omega)|_{\omega=\tilde{\omega}_k} = -[\operatorname{res}\mathbf{u}(\omega)|_{\omega=\omega_k}]^*$ and $e^{-i\omega_k t} = (e^{-i\tilde{\omega}_k t})^*$ and assumed that the poles ω_k are simple and are not on the imaginary axis. If there are poles on the imaginary axis, the contour L must be modified so that they remain outside it (their contribution is complex-valued, which is unacceptable for a real transient solution $\mathbf{v}_1(t)$).

†LABREUCHE, C., Inverse obstacle scattering based on resonant frequencies. *INRIA Conference on Inverse Problems of Wave Propagation and Diffraction*, Aix-les-Bains, France, 23–27 September 1996.

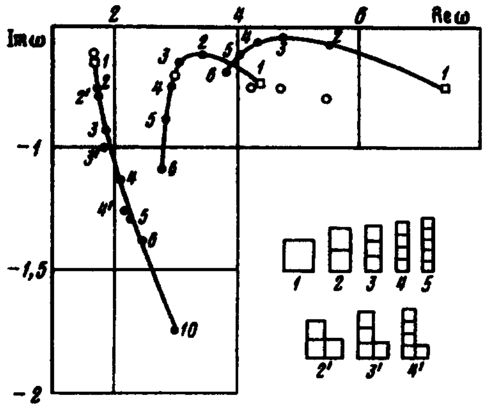

Fig. 1.

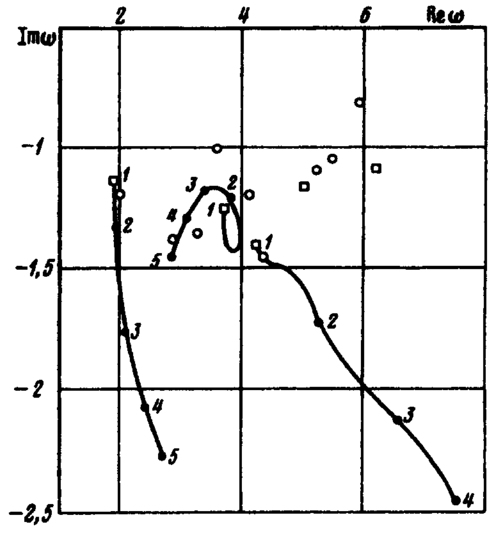

Fig. 2.

The sum of the residues in (4) is the set of exponentially decaying harmonic oscillations $\mathbf{v}_k(\mathbf{x}, t) = \{v_k^{(1)}, v_k^{(2)}, v_k^{(3)}\}$ of the form

$$v_k^{(l)}(\mathbf{x},t) = 2a_k^{(l)}(\mathbf{x})e^{-\beta_k t}\sin(\theta_k^{(l)} - \alpha_k t) \tag{5}$$

where $\alpha_k^{(l)}$, $\theta_k^{(l)}$ are the modulus and argument of the complex components of the residue: res $u^{(l)}|_{\omega=\omega_k} = \alpha_k^{(l)}e^{i\theta_k^{(l)}}$, and α_k, β_k are the real and imaginary parts of the poles: $\omega_k = \alpha_k - i\beta_k$; α_k, $\beta_k > 0$, giving respectively the angular frequency and decay decrement of these harmonics.

In practical non-destructive testing, signals of the form (5), which are emitted by cracks, are recorded by acoustic emission methods.

In the case of a homogeneous isotropic space, the structure of the matrix-kernel of the integral equations is such that the system can in fact be split into two independent systems: in terms of the normal and tangential components of the displacement jump (cf. (1.4) [1]). Accordingly, system (1) also splits into two independent systems and $\Delta(\omega) = \Delta_1(\omega)\Delta_2(\omega)$, where $\Delta_1(\omega)$, $\Delta_2(\omega)$ are the determinants of these systems for the normal and tangential components.

Thus, for an isotropic homogeneous medium, the set of resonance poles consists of two independently determined sets $\omega_k^{(1)}$ and $\omega_k^{(2)}$ (the roots of the functions Δ_1 and Δ_2), which we call poles of the first and second types [9]. The reflected field usually carries resonances of both kinds, but in the special case where a *P*- or *S*-wave is normal to the crack, *P*- or *S*-waves with resonant frequencies of only the first or the second type, respectively, are also reflected. There is no such splitting when the cracks are on the surface of a joint between two materials of different kinds (or, in the general case, in vertically inhomogeneous media), but in this case too one can nominally identify the poles as being of the first or second type, tracking their correspondence as the properties of the materials change continuously from the composite to the homogeneous half-space.

A numerical search was made for resonance poles in the complex plane by Müller's method of parabolas, the number of roots of the analytic functions $\Delta_1(\omega)$, $\Delta_2(\omega)$ being found by the principle of the argument. The results were checked by comparing them with the known results for a circular crack [8, 9].

Figures 1 and 2, respectively, show the results for poles of the first and second types in the complex plane of the dimensionless frequency $\omega = 2\pi f r_0/\upsilon_S$ (f is the dimensional frequency, υ_S is the velocity of *S*-waves, $r_0 = \sqrt{(S_\Omega/\pi)}$ is the characteristic linear dimension and S_Ω is the area of the crack, for a circular region r_0 is equal to its radius); Poisson's ratio $\nu = 0.25$. The open circles are the resonance poles for a circular crack, obtained previously,† the small squares correspond to a square crack of the same area. There are more open circles because removable poles are included for the circular crack. The non-removable resonant frequencies for a circle and a square are quite close. The main factor here is the area of the crack. We note that for two similar regions Ω_1, Ω_2, differing only in

†ALVES, C., Étude numérique de la diffraction d'ondes acoustiques et élastiques par une fissure plane de forme quelconque. Problèmes directs et inverses. Doctoral thesis, L'École Polytechnique, Compiègne, France, 1995.

their area S_1 and S_2, the pole distribution patterns are also similar, with similarity factor $K = \sqrt{(S_1 S_2)}$: $\omega^{(2)} = k\omega^{(1)}$. By virtue of the frequency normalization chosen above, the area of any region Ω in dimensionless units (relative to r_0) is equal to the area of the unit circle $S_0 = \pi$, and this is convenient for analysing the effect of the shape.

Figures 1 and 2 show the trajectory of motion of the first three non-removable poles of the first and second types as the region changes continuously from a square to an elongated rectangle. The numbers 1, 2, ... indicate the position of the poles for rectangular regions with a ratio of the sides of 1:1, 1:2, etc. Lengthening the crack greatly affects the pole distribution, whereas it differs very little in the case of a square and a circle. It is also worth noting that the change in the position of the first pole of the first type as a function of the eccentricity of the elliptical cracks [9] agrees with the path of displacement for rectangular cracks downwards and to the right (Fig. 1).

Apart from rectangles, we considered L-shaped cracks (the shapes are shown in the bottom right-hand part of Fig. 1). The protuberance has very little effect on the position of the pole (see points 2′ and 4′ in Fig. 1, for instance). The main factor influencing the position of the resonance poles appears to be the ratio of the length and the main transverse dimension.

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